An introduction to the space of real-analytic modular forms.

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Young Scholars in the Analytic Theory of Numbers and Automorphic Forms

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1. What is a real-analytic modular form?

Definition

- Let $r, s \in \mathbb{Z}$, $\Gamma = SL(2, \mathbb{Z})$.

Definition

A real-analytic function $F : \mathcal{H} \to \mathbb{C}$ is a real-analytic modular form of weights (r, s) if

$$(i) \quad F(\gamma(z)) = (cz + d)^r (c\bar{z} + d)^s F(z), \quad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

(ii) $F(z)$ has the Fourier expansion of the form

$$F(z) = \sum_{|j| \le M} y^j \left(\sum_{m,n \ge -N} a_{m,n}^{(j)}\, e^{2\pi i m z} e^{-2\pi i n \bar{z}} \right)$$

where $z = x + iy$ and $M, N \in \mathbb{N}$.

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where $z = x + iy$ and $M, N \in \mathbb{N}$.

- $\mathcal{M}^! := \bigoplus_{r,s} \mathcal{M}^!_{r,s}$.
- Introduced and studied by Brown.[1]

[1] F. Brown, "A class of non-holomorphic modular forms I," Res. Math. Sci., vol. 5, no. 7, 2018.

Motivation

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1. Contains or intersects various spaces of important modular objects.
    - Classical modular forms,
    - Weakly anti-holomorphic modular forms,
    - Maass forms.

   Consider real-analytic modular forms as a unifying tool for these spaces.

2. Help understand modular graph forms appearing in string theory in physics.

Examples

- $\mathbb{L} := i\pi(z - \bar{z}) = -2\pi y \quad \in \mathcal{M}^!_{-1,-1}$.
- For $r \geq 4$, the Eisenstein series

$$\mathbb{G}_r(z) = -\frac{B_r}{2r} + \sum_{n=1}^{\infty} \sigma_{r-1}(n) e^{2\pi i n z},$$

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The real-analytic Eisenstein series of weights (r, s)

For $r, s \geq 0$ and $r + s = w \geq 2$, we have

$$\mathcal{E}_{r,s} = \frac{w!}{2 \cdot (2\pi i)^{w+2}} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{\mathbb{L}}{(mz+n)^{r+1}(m\bar{z}+n)^{s+1}}.$$

Maass operators and the Laplacian

The operators

$$\partial_r : \mathcal{M}^!_{r,s} \to \mathcal{M}^!_{r+1,s-1} \quad \text{and} \quad \overline{\partial}_s : \mathcal{M}^!_{r,s} \to \mathcal{M}^!_{r-1,s+1}$$

are given by

$$\partial_r = 2iy\frac{\partial}{\partial z} + r \quad \text{and} \quad \overline{\partial}_s = -2iy\frac{\partial}{\partial \bar{z}} + s.$$

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The Laplacian $\Delta_{r,s} : \mathcal{M}^!_{r,s} \to \mathcal{M}^!_{r,s}$

$$\Delta_{r,s} = -\overline{\partial}_{s-1}\partial_r + r(s-1) = -\partial_{r-1}\overline{\partial}_s + s(r-1).$$

Usually just write Δ.

2. L-functions for real-analytic modular forms

L-functions

- Can define L-functions for the entirety of $\mathcal{M}^!$.
- The Fourier expansion

$$F(z) = \sum_{|j|\leq M} y^j \sum_{m,n\geq -N} a^{(j)}_{m,n}\, q^m \bar{q}^n,$$

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$$\tilde{f}(z) := \sum_{|j|\leq M} y^j \sum_{\substack{m,n\geq -N \\ m+n>0}} a_{m,n}^{(j)} q^m \bar{q}^n,$$

$$f^0(z) := \sum_{|j|\leq M} y^j \sum_{\substack{m,n\geq -N \\ m+n=0}} a_{m,n}^{(j)} q^m \bar{q}^n,$$

$$\mathring{f}(z) := \sum_{|j|\leq M} y^j \sum_{\substack{m,n\geq -N \\ m+n<0}} a_{m,n}^{(j)} q^m \bar{q}^n.$$

L-functions

For a real-analytic modular form $f \in \mathcal{M}^!_{r,s}$, the L-function is given by (for $v \neq -j,\ r+s+j\ (|j| \leq M)$) [2]

$$L_f^*(v) := \left(\int_1^{\infty} \tilde{f}(it) t^{v-1} dt + \int_1^{-\infty} \mathring{f}(it) t^{v-1} dt - \sum_{|j| \leq M} \sum_{\substack{m,n \geq -N \\ m+n=0}} \frac{a_{m,n}^{(j)}}{v+j} \right)$$

$$+ i^{r-s} \left(\int_1^{\infty} \tilde{f}(it) t^{r+s-v-1} dt + \int_1^{-\infty} \mathring{f}(it) t^{r+s-v-1} dt \right.$$

$$\left. - \sum_{|j| \leq M} \sum_{\substack{m,n \geq -N \\ m+n=0}} \frac{a_{m,n}^{(j)}}{r+s-v+j} \right).$$

[2] N. Diamantis and J. Drewitt, "Period functions associated to real-analytic modular forms," Res. Math. Sci., vol. 7, no. 21, 2020.

L-functions

- When restricting to subspaces, it matches previously given L-functions.

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- In the case of weakly holomorphic modular forms:

$$L_f^*(v) = \sum_{\substack{m \geq m_0 \\ m \neq 0}} \frac{a_m^{(0)} \Gamma(v, 2\pi m)}{(2\pi m)^v} + i^k \sum_{\substack{m \geq m_0 \\ m \neq 0}} \frac{a_m^{(0)} \Gamma(k - v, 2\pi m)}{(2\pi m)^{k-v}} - b\left(\frac{1}{v} + \frac{i^k}{k - v}\right),$$

which matches previous literature. [3]

[3] K. Bringmann, N. Diamantis, S. Ehlen, "Regularized inner products and errors of modularity," Int. Math. Res. Not., vol. 2017, no. 24, 2017.

3. The space of modular iterated integrals

Modular iterated integrals

- $\mathcal{MI}^!_{-1} = 0$.
- For $n \geq 0$, $\mathcal{MI}^!_n$ is largest subspace of $\bigoplus_{r,s\geq 0} \mathcal{M}^!_{r,s}$ which satisfies

$$\partial \mathcal{MI}^!_n \subset \mathcal{MI}^!_n + M^![\mathbb{L}] \otimes \mathcal{MI}^!_{n-1},$$
$$\overline{\partial} \mathcal{MI}^!_n \subset \mathcal{MI}^!_n + \overline{M}^![\mathbb{L}] \otimes \mathcal{MI}^!_{n-1},$$

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where $M^!$ is the space of weakly holomorphic modular forms.

$$\mathcal{MI}^!_0 \subset \mathcal{MI}^!_1 \subset \mathcal{MI}^!_2 \subset \cdots \subset \mathcal{MI}^!.$$

Length two

- $r, s, p, q \geq 0,\ r + s \geq 2,\ p + q \geq 2$

$$\mathcal{E}_{r,s}\,\mathcal{E}_{p,q} \in \mathcal{MI}^!_2.$$

- [Brown][4] discovered certain functions denoted by $(F^{(k)}_{2a,2b})_{r,s}$.

[4] F. Brown, "A class of non-holomorphic modular forms I," Res. Math. Sci., vol. 5, no. 7, 2018.

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- These functions have Laplace-eigenvalue equations

$$(\Delta + 2)(\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2}) = -\mathbb{L}^4\mathbb{G}_4\overline{\mathbb{G}}_4 - \mathbb{L}^2\mathcal{E}_{1,1}\mathcal{E}_{1,1},$$

$$(\Delta + 2)(F^{(1)}_{2,2})_{2,0} = -4\mathbb{L}^2\mathbb{G}_4\mathcal{E}_{0,2},$$

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$$(\Delta + 2)(F^{(1)}_{2,2})_{1,1} = -4\mathbb{L}^3\mathbb{G}_4\overline{\mathbb{G}}_4.$$
- [Dorigoni, Kleinschmidt, Schlotterer][5] defined functions $A^{+(t)}_{m,n}$ and $A^{-(t)}_{m,n}$, which play an important role in the theory of depth-two modular graph forms.

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[4] F. Brown, "A class of non-holomorphic modular forms I," Res. Math. Sci., vol. 5, no. 7, 2018.

[5] D. Dorigoni, A. Kleinschmidt, and O. Schlotterer, "Poincaré series for modular graph forms at depth two. I. Seeds and Laplace systems." J. High Energy Phys., vol. 2022, no. 133, 2022.

Length two

We have

$$(\Delta+2)A^{+(2)}_{2,2} = (\Delta+2)\left(16\mathbb{L}^2\mathcal{E}_{2,0}\mathcal{E}_{0,2} - 4\mathbb{L}(F^{(1)}_{2,2})_{1,1}\right),$$

$$(\Delta+6)A^{+(3)}_{2,3} = (\Delta+6)\left(\frac{4}{3}\mathbb{L}^3\mathcal{E}_{2,0}\mathcal{E}_{1,3} - \frac{2}{9}\mathbb{L}^2(F^{(1)}_{2,4})_{2,2}\right),$$

and

$$(\Delta+2)A^{-(2)}_{2,3} = (\Delta+2)\left(4\mathbb{L}^3\mathcal{E}_{2,0}\mathcal{E}_{1,3} - \frac{4}{3}\mathbb{L}^3\mathcal{E}_{1,1}\mathcal{E}_{2,2} - \frac{4}{3}\mathbb{L}(F^{(2)}_{2,4})_{1,1}\right).$$

- Currently researching the underlying connection between these spaces [J., Schlotterer, Kleinschmidt, Matthes, Doroudiani, Hidding, Verbeek].

Length three?

- What about length three?
- Can find analogous functions to $(F^{(k)}_{2a,2b})_{r,s}$ for the length three case:[6]

$$(\Delta + 8)(G^{(0)}_{4,2,2})_{8,0} = -\mathbb{L}\mathbb{G}_6 \mathcal{E}_{2,0}\mathcal{E}_{1,1},$$
$$(\Delta + 8)(G^{(0)}_{4,2,2})_{7,1} = -\mathbb{L}\mathbb{G}_6 \mathcal{E}_{1,1}\mathcal{E}_{1,1} - 2\mathbb{L}\mathbb{G}_6 \mathcal{E}_{2,0}\mathcal{E}_{0,2},$$

[6] J. Drewitt, "Laplace-eigenvalue equations for length three modular iterated integrals," J. Number Theory. (In press).

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- May have similar connections between length three modular iterated integrals and depth-three modular graph forms.

[6] J. Drewitt, "Laplace-eigenvalue equations for length three modular iterated integrals," J. Number Theory. (In press).

Thank you!